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Kinetic Limit of N-Body Description of Wave–Particle Self-Consistent Interaction

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A system of N particles $\xi^N = (x_1, v_1, ..., x_N, v_N)$ interacting self-consistently with one wave $Z = A \exp(i\phi)$ is considered. Given initial data $(Z^{(N)}(0), \xi^N(0))$, it evolves according to Hamiltonian dynamics to $(Z^{(N)}(t), \xi^N(t))$. In the limit $N \to \infty$, this generates a Vlasov-like kinetic equation for the distribution function $f(x, v, t)$, abbreviated as $f(t)$, coupled to the envelope equation for Z: initial data $(Z^{(\infty)}(0), f(0))$ evolve to $(Z^{(\infty)}(t), f(t))$. The solution (Z, f) exists and is unique for any initial data with finite energy. Moreover, for any time $T > 0$, given a sequence of initial data with N particles distributed so that the particle distribution $f^N(0) \to f(0)$ weakly and with $Z^{(N)}(0) \to Z(0)$ as $N \to \infty$, the states generated by the Hamiltonian dynamics at all times $0 \leqslant t \leqslant T$ are such that $(Z^{(N)}(t), f^N(t))$ converges weakly to $(Z^{(\infty)}(t), f(t))$.

KEY WORDS: Plasma; kinetic theory; wave–particle interaction; mean-field limit.

1. INTRODUCTION

Recent work on the dynamics of wave-particle interaction has led to extensive use of N-body Hamiltonian models in parallel with the more traditional kinetic approach. The present paper aims at discussing to what extent the two approaches agree in the limit $N \to \infty$, where N-body dynamics formally reduces to kinetic theory. This is a classical problem of statistical physics, which is only partly solved for particles interacting through short-range forces: significant results in this program are the derivation of the Boltzmann equation from the Liouville equation in the Boltzmann–Grad limit in the pioneering work of Lanford and King,

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limited to short timescales,[1,2] and more recent advances.[3,4] However, for long-range forces, and more precisely for smooth enough mean-field interactions, the formal limit $N \to \infty$ commutes with the dynamics.[5,6] We show in this paper how the mean-field methods apply also to wave-particle interactions.

A physical motivation for this work is that wave-particle interacting systems are typical of plasmas and common to many physical phenomena. The paradigm of such interactions is provided by the self-consistent Hamiltonian $H_{sc}^{N,M}$ describing the evolution of N particles and M Langmuir waves.[7–20] In particular this Hamiltonian enables a unified mechanical approach of classical plasma problems like Landau damping and beam-plasma instability, by treating Langmuir waves as M harmonic oscillators self-consistently coupled to N quasiresonant beam particles.

More complex Hamiltonians are used in the modeling of laser-plasma interactions, free electron lasers and beams in traveling wave tubes (see, e.g., refs. 21–24 and references therein). Such systems are typically described by the Maxwell equations for the fields, coupled with the Vlasov equation for the distribution of particles, the motion of individual particles being computed a posteriori from the obtained evolving fields. As the modal description of the fields brings the model of the system to a form similar to the Hamiltonian (1) below, our results may be extended to these cases, assessing the validity of common practice.

Let us now point out the major motive for this work.

Increasing computing capacities have recently made it possible to investigate numerically the long-time fate of plasmas, feeding a controversial debate in the kinetic approach, with numerical and theoretical arguments.[25–27] In those simulations starting from smooth initial data $f(x, v)$, one commonly observes that complicated distribution functions are generated, with long thin tendrils in particle (x, v)-space Λ on scales eventually smaller than any interparticle distance in the corresponding N-body description. This phenomenon is well measured by the growth of a numerical entropy like $-\int_\Lambda f \ln f \, dx \, dv$ or of the entropylike functional $-\int_\Lambda f^2 \, dx \, dv$[27] in all kinetic simulations, though these quantities are invariant for the mathematical solution of the Vlasov equation. This phenomenon may be connected to the fact that simulations use codes that necessarily induce a discretized treatment of the phase space. Besides, physical systems are obviously discrete as they only involve a finite number of particles. A related process, the formation of (x, v)-space granulations, has been invoked as an important process in the dynamics of runaway particles in hot (collisionless) plasmas; these particles' dynamics has been analyzed in a mixed approach, combining kinetic theory and classical mechanics.[28,29] All this naturally questions the relevance of kinetic results and provides a fundamental reason for investigating the compatibility between the kinetic limit $N \to \infty$ and the time evolution of the wave-particle system.

The basic characteristic of the wave-particle models is that particles do not interact directly with each other: they only interact with the modes; symmetrically, the modes do not interact directly with each other: they only interact directly with the particles. This type of coupling is characteristic of weak turbulence. Inasmuch the modes are spatial Fourier components of some fields, these components are not localized spatially: this invites to describe the many-body limit $N \to \infty$ as a mean-field limit and enables us to apply the techniques which succeed in the case of particle-particle mean-field coupling.

The present work takes advantage of this observation to show that the kinetic limit $N \to \infty$ and the time evolution over any time interval $[0, T]$ commute. Our result implies (through equations (32), (33) and the corollary of the theorem below) that numerical simulations with increasing number of particles behave ever closer to the predictions of kinetic theory, and that the dynamics of "test particles" in the self-consistent fields converges in a weak sense to the actual dynamics of the particles in the full many-body system. For simplicity we present our results in one space dimension with periodic boundary conditions, which conforms to the physical conditions considered in models of plasmas as implied, e.g., by spatial confinement.[30–32]

In Section 2 we describe the model and its evolution equations. The main results are stated in Section 3. Section 4 is devoted mainly to a finite N estimate and a technical remark, preparing the proof presented in Section 5. The final section is devoted to the conclusion.

2. SELF-CONSISTENT HAMILTONIAN AND KINETIC LIMIT

We first consider a general system of N particles with respectively mass m_r, charge q_r, position x_r and momentum p_r, interacting with M waves with respectively natural frequency ω_{j0}, phase θ_j and intensity I_j. The evolution of this system is described by the Hamiltonian

$$H_{sc}^{N,M} = \sum_{r=1}^{N} \frac{p_r^2}{2m_r} + \sum_{j=1}^{M} \omega_{j0} I_j - \varepsilon \sum_{r=1}^{N} \sum_{j=1}^{M} q_r k_j^{-1} \beta_j \sqrt{2I_j} \cos(k_j x_r - \theta_j) \qquad (1)$$

where the first term corresponds to free particles, the second term to free waves (harmonic oscillators) and the third term to their coupling. The coupling constants are expressed in such a way to ease the kinetic limit $N \to \infty$: we shall keep the "wave susceptibilities" β_j constant in this limit.

A simple change of variables enables one to ensure that all coefficients $\beta_j > 0$, which is assumed in the following. The overall coupling factor ε in the interaction term of (1) recalls our interest in the weak-coupling regime ($\varepsilon \ll 1$),[8, 20] though its smallness is irrelevant in this paper.

Assuming periodic boundary conditions, the particles move on $(\mathbb{R}/L\mathbb{Z}) = S_L$ and the wavenumbers are quantized ($k_j = n_j 2\pi/L$ for some integer n_j); the extension to several space dimensions is easy.[33] The phase space of this system is thus $(S_L \times \mathbb{R})^N \times \mathscr{Z}^M$ where $\mathscr{Z} = S_{2\pi} \times \mathbb{R}^+$ for each mode. The self-consistent dynamics generated by H_{sc} admits two obvious constants of the motion: total energy $E = H_{sc}$ and total momentum $P = \sum_r p_r + \sum_j k_k I_j$.

The natural scaling of our model in the limit $N \to \infty$ is easily deduced from its equilibrium (Gibbs) thermodynamics. Then the energy $E = H_{sc}$ and the wave intensities I_j are extensive (i.e., $O(N)$), and the coupling constant scales as $\varepsilon = O(N^{-1/2})$. The extensivity of wave intensities can easily be interpreted as, in the physical regime of the model, we expect particles to be mostly resonant with the waves, each such particle contributing then to wave intensities by evolving in their potential well. This prompts us to introduce intensive wave envelopes

$$a_j = N^{-1/2} \sqrt{2I_j}\, e^{-i\theta_j + i\omega_{j0} t} \tag{2}$$

and renormalized coupling constants $\beta'_j = N^{1/2} \varepsilon \beta_j$.

To simplify calculations and notations, we introduce other non-canonical variables, namely particle velocities $v_r = p_r/m_r$ and restrict ourselves to a single species (all $q_r = q > 0$, $m_r = m$) and only one mode ($M = 1$, all indices j will be omitted henceforth), the extension of the forthcoming results to the general case being straigthforward.[33]

The evolution equations now become

$$\dot{x}_r = v_r \tag{3}$$

$$\dot{v}_r = \frac{iq}{2m} \beta' (a e^{ikx_r - i\omega_0 t} - a^* e^{-ikx_r + i\omega_0 t}) \tag{4}$$

$$\dot{a} = \frac{i}{N} \beta' k^{-1} \sum_{r=1}^{N} q e^{-ikx_r + i\omega_0 t} \tag{5}$$

The usual space of kinetic theory is Boltzmann's μ-space $\Lambda = S_L \times \mathbb{R}$. The positions and velocities of the N particles determine a sum of point measures σ^N on Λ:

$$\sigma^N(x, v, t) = \frac{1}{N} \sum_{r=1}^{N} \delta(x - x_r(t))\, \delta(v - v_r(t)) \tag{6}$$

with unit total mass irrespective of the number N of particles. The **kinetic limit**, formally $N \to \infty$, corresponds to considering a sequence of N-particle point measures σ^N converging to a measure σ, defined by a positive continuously differentiable density $f(x, v, t)$ w.r.t. Lebesgue measure in Λ, in the weak sense for a natural space of test functions $\mathscr{D}$. Denote by $\mathscr{F}$ the space of positive normalized measures μ on Λ with finite momentum and kinetic energy, i.e., such that $\int d\mu = 1$, $\int v^2 \, d\mu < \infty$ and define on $\mathscr{F}$ the bounded-Lipshitz distance

$$d_{bL}(\mu, \mu') \equiv \sup_{\phi \in \mathscr{D}} \left| \int_\Lambda \phi \, d\mu - \int_\Lambda \phi \, d\mu' \right| \tag{7}$$

with the set of bounded, Lipschitz-continuous normalized test functions

$$\begin{aligned} \mathscr{D} \equiv \{ \phi : \Lambda \to [0, 1], \; &|\phi(x, v) - \phi(x', v')| \\ &\leqslant \|(x, v) - (x', v')\| \; \forall (x, v), (x', v') \in \Lambda \} \end{aligned} \tag{8}$$

Here Λ is equipped with the distance $\|(x, v) - (x', v')\| \equiv \alpha(d_L(x, x') + \tau \, |v - v'|)$, where α^{-1} and τ are respectively convenient length and time scales to be chosen below and d_L stands for the absolute length of the minimal arc connecting two points on the circle S_L, that is $d_L(x, x') = \min_{k \in \mathbb{Z}} |x - x' + kL|$.

Then we consider the distance on $\mathscr{Z}$,

$$\|a - a'\| = w \, |a - a'| \tag{9}$$

where the real positive coefficient w will be chosen below in (25), and $|a|$ is the modulus of the complex number a. Our distance on $\mathscr{F} \times \mathscr{Z}$ is just

$$\|(\mu, a) - (\mu', a')\| = d_{bL}(\mu, \mu') + \|a - a'\| \tag{10}$$

Finally, consider a time-dependent pair $(\mu_t, a(t))$ where μ_t stands for an absolutely continuous measure with density $f(x, v, t)$. The kinetic model dual to (3)–(5) is a system coupling a Vlasov-like equation for $f(x, v, t)$ with a source equation for $a(t)$:

$$\partial_t f + v \, \partial_x f + \frac{iq}{2m} \beta'(a e^{ikx - i\omega_0 t} - a^* e^{-ikx + i\omega_0 t}) \, \partial_v f = 0 \tag{11}$$

$$\dot{a} = iq\beta' k^{-1} \int_\Lambda f(x, v, t) \, e^{-ikx + i\omega_0 t} \, dx \, dv \tag{12}$$

This dynamics leaves $\mathscr{F} \times \mathscr{L}$ invariant. For finite N the point measure (6) and envelope associated to the particles and mode evolving according to (3)–(5) form a weak solution of the system (11), (12).

3. MAIN RESULTS

The self-consistent dynamics (3)–(5) preserves two constants of the motion, namely total energy H and total momentum $P = \sum_r p_r + kI$. In the kinetic limit, we consider the normalized constants $h = H/N$ and $p = P/N$:

$$h(\mu, a) = \int_\Lambda \frac{mv^2}{2} d\mu(x, v) + \omega_0 \frac{|a|^2}{2} - \int_\Lambda qk^{-1}\beta' \Re(ae^{ikx - i\omega_0 t})\, d\mu(x, v) \tag{13}$$

$$p(\mu, a) = \int_\Lambda mv\, d\mu(x, v) + k\frac{|a|^2}{2} \tag{14}$$

where $\Re$ denotes the real part. For any finite N and h, the energy surface $H_{sc}^{N,1} = Nh$ in $\Lambda^N \times \mathscr{L}$ is compact, and the vector field (3)–(5) is continuous and bounded on it. This ensures that the dynamics generates a group for all initial conditions.

Moreover, the first variation of the dynamics (3)–(5) generates a linear operator $\mathscr{M} = \partial(\dot{x}_r, \dot{v}_r, \dot{a})/\partial(x_r, v_r, a)$, depending continuously on (x_r, v_r, a). As the energy surface is compact for any given N, $\mathscr{M}$ is bounded. With the specific form of H_{sc}, we show that, with appropriate choice of the constant w:

$$\|\mathscr{M}\| \leqslant \tau^{-1} + \gamma[a(t)] \tag{15}$$

where

$$\tau = \left(\frac{q^2\beta'^2}{m}\right)^{-1/3} \tag{16}$$

$$\gamma[a(s)] = \frac{q\tau}{m} \beta' k\, |a(s)| \tag{17}$$

The positive function $\gamma[a(s)]$ is continuous on the energy surface, on which it has an upper bound uniform with respect to N.

The kinetic limit, $N \to \infty$, admits a similar bound, ensuring the existence and uniqueness

Theorem. Given initial data $(\mu_0, a(0)), (\mu'_0, a'(0)) \in \mathscr{F} \times \mathscr{Z}$, with $h_0 = h(\mu_0, a(0))$ and $h'_0 = h(\mu'_0, a'(0))$, the kinetic evolution equations generate for all times $t \geqslant 0$ states $(\mu_t, a(t))$ and $(\mu'_t, a'(t))$ respectively from these data. Moreover,

$$d_{bL}(\mu_t, \mu'_t) + \|a(t) - a'(t)\| \leqslant e^{Ct}(d_{bL}(\mu_0, \mu'_0) + \|a(0) - a'(0)\|) \qquad (18)$$

for some $C \leqslant c_1 + c_2 |h_0|^{1/2} < \infty$, with c_1 and c_2 two strictly positive constants independent of initial data.

This theorem implies the

Corollary. Given a continuous measure $\sigma_0 \in \mathscr{F}$ and a sequence of discrete probability measures $\sigma_0^N \in \mathscr{F}$ defining the initial distribution of particles in (x, v) space, such that $\lim_{N \to \infty} d_{bL}(\sigma_0^N, \sigma_0) = 0$, given an initial wave envelope $a(0) \in \mathscr{Z}$, and given any time $T > 0$, consider for all $0 \leqslant t \leqslant T$ the resulting measures and wave envelopes $(\sigma_t^N, a^N(t))$ generated by $H_{sc}^{N,1}$ and the kinetic solution $(\sigma_t = f(x, v, t)\, dx\, dv, a^\infty(t))$ of (11), (12). Then $\lim_{N \to \infty} d_{bL}(\sigma_t^N, \sigma_t) = 0$ and $\lim_{N \to \infty} a^N(t) = a^\infty(t)$, uniformly on $[0, T]$.

4. PRELIMINARY REMARKS

For given N and finite energy H, the first variation $\mathscr{M}$ of the dynamics (3)–(5) has bounded norm (with the L_1 distance):

$$\begin{aligned}\|(\delta\dot{x}, \delta\dot{v}, \delta\dot{a})\|_1 &\equiv N^{-1} \sum_{r=1}^{N} \alpha(|\delta\dot{x}_r| + \tau\, |\delta\dot{v}_r|) + w\, |\delta\dot{a}| \\ &= N^{-1} \sum_{r=1}^{N} \|(\delta\dot{x}_r, \delta\dot{v}_r)\| + w\, |\delta\dot{a}| \qquad (19)\end{aligned}$$

We readily find

$$|\delta\dot{x}_r| = |\delta v_r| \qquad (20)$$

$$|\delta\dot{a}| = N^{-1} \left| \sum_{r=1}^{N} \beta' q e^{ikx_r - i\omega_0 t}\, \delta x_r \right| \leqslant N^{-1} \beta' q \sum_{r=1}^{N} |\delta x_r| \qquad (21)$$

and

$$|\delta\dot{v}_r| = \left| \frac{q\beta'}{m} \Re(e^{ikx_r - i\omega_0 t}(ak\, \delta x_r - i\, \delta a)) \right| \qquad (22)$$

$$\leqslant \frac{q\beta'}{m} k\, |a| \cdot |\delta x_r| + \frac{q\beta'}{m} \cdot |\delta a| \qquad (23)$$

so that

$$\|(\delta\dot{x}, \delta\dot{v}, \delta\dot{a})\|_1 \leqslant N^{-1} \sum_{r=1}^{N} \alpha\tau^{-1}(\tau\gamma[a(t)] \cdot |\delta x_r| + \tau\,|\delta v_r|) + wN^{-1}\beta' q \sum_{r=1}^{N} |\delta x_r| + \alpha\tau \frac{q\beta'}{m} \cdot |\delta a| \tag{24}$$

with $\gamma[a(s)]$ defined by (17). The four causes for the divergence of trajectories in $\Lambda^N \times \mathscr{X}$ are saddle points (in (x, v) plane) associated with maxima of the modes' potentials (the $|a|$ contribution to $\gamma[a(t)]$), velocity shear (the velocity term), the dependence of the modes source on the particle positions, and the dependence of the saddle points themselves on the mode envelopes.

An appropriate choice of constants α, τ, w keeps the estimates as small as possible. Thus let

$$w = \alpha w_0 \beta' \tag{25}$$

and solve

$$\tau^{-1} = w_0 q \beta'^2 = \frac{q\tau}{m w_0} \tag{26}$$

This leads to the expression of τ announced in (16) and to

$$w_0 = (qm)^{-1/3} \beta'^{-4/3} \tag{27}$$

so that (24) reduces to

$$\|(\delta\dot{x}, \delta\dot{v}, \delta\dot{a})\| \leqslant \tau^{-1} \|(\delta x, \delta v, \delta a)\| + \gamma[a(t)]\, N^{-1} \sum_r \alpha\, |\delta x_r| \tag{28}$$

which implies (15). Constant α remains arbitrary, as it only determines the scale of the distances in Λ and $\mathscr{X}$, and (15) is homogeneous (degree 1). Considering only the restricted dynamics on Λ, with $\delta a = 0$, (24) straightforwardly leads to the continuity equation

$$\|(\delta\dot{x}, \delta\dot{v})\| \leqslant \gamma'[a(t)]\, \|(\delta x, \delta v)\| \tag{29}$$

with

$$\gamma'[a(t)] = \max(\tau^{-1}, \gamma[a(t)]) \tag{30}$$

Note that $\gamma'[a(t)]$ is bounded uniformly in time, as the positive function $\gamma[a]$ is bounded above on the energy surface by a function which does not grow faster than $h^{1/2}$ in the large energy limit. More precisely, let $\lambda > 0$ solve $\lambda^2 \omega_0^{-1}(q\beta' k/m)^2 = 2h + \omega_0^{-1}(q\beta'/k)^2$. Then $|\beta' kaq/m| \leqslant (q^2/m)\,\omega_0^{-1}\beta'^2 (1 + k^2\lambda/m)$. It should be noted once more that (29) reflects that the divergence rate in (x, v) is controlled by velocity shear and by saddle points of the pendulum-like potential depending on wave amplitudes. The latter situation typically corresponds to a trapping regime for large enough wave intensities.

Finally, note the following

Proposition 1. Let $Y\colon \Lambda \to \Lambda$ be a Lipschitz mapping with constant $L \geqslant 1$ on Λ, and $\mu, \nu \in \mathscr{F}$. Then:

$$\sup_{\phi \in \mathscr{D}} \left| \int_\Lambda \phi \circ Y d(\mu - \nu) \right| \leqslant L d_{bL}(\mu, \nu) \tag{31}$$

Proof. Clearly $L^{-1}\phi \circ Y \in \mathscr{D}$ for any $\phi \in \mathscr{D}$. Hence $\sup_{\phi \in \mathscr{D}} |\int_\Lambda L^{-1}\phi \circ Yd(\mu - \nu)| \leqslant d_{bL}(\mu, \nu)$.

5. PROOF OF THE MAIN RESULT

The proof of Theorem 1 uses the fact that the two types of degrees of freedom have no "self"-interaction, which greatly simplifies our Cauchy problems. Concerning the N-body Cauchy problem consisting in equations (3)–(5) and given initial conditions in $(S_L \times \mathbb{R})^N \times \mathscr{Z}$, there exists obviously a unique solution global in time. The motion of particle r is completely determined by its initial position and velocity and by the mode history, i.e., the data of the envelope $a(.)$ over a time interval $[s, t]$ defines the vector field G so that:

$$\frac{d}{dt}(x_r(t), v_r(t)) = G[a(t)](x_r(t), v_r(t)) \tag{32}$$

Let us now express the Cauchy problem in terms of measures for any given initial pair $(\mu_0, a(0))$. The evolution in the one-particle space Λ reads also:

$$\frac{d}{dt}(x(t), v(t)) = G[a(t)](x(t), v(t)) \tag{33}$$

Assuming the existence of a solution $(\mu_t, a(t))$ weakly continuous w.r.t. time, then this vector field is continuous w.r.t. time, as, by construction, $a(t)$ is then well defined and even derivable. It is Lipschitz-continuous on Λ according to (29) and subsequent remarks. Thus Cauchy–Lipschitz theorem ensures the existence and unicity of the flow T:

$$(x_r(t), v_r(t)) = T_{t,s}[a(.)](x_r(s), v_r(s)) \tag{34}$$

By duality the point measure μ_s on Λ is transported by the flow to

$$\mu_t = \mu_s \circ T_{s,t}[a(.)] \tag{35}$$

Similarly, the evolution of the mode is also completely determined by its initial data $a(s)$ and by the history of the measure on one-particle space Λ which defines a flow S by

$$a(t) = S_{t,s}[\mu_.]\, a(s) \tag{36}$$

Solving kinetic equations with initial data $(\mu_0, a(0))$ amounts then to finding in the space $(\mathscr{F} \times \mathscr{Z})^{\mathbb{R}}$ a fixed point of the coupled system (35), (36) that expresses in terms of measures equations (11), (12). Our strategy now follows and extends in some way that of Neunzert[5] and Spohn,[6] who considered direct particle-particle interaction of mean-field type.

We shall first prove the uniqueness of the (hypothetic) solution starting from given initial data, then establish the existence of a weakly continuous $(\mu_t, a(t))$, solution to system (35), (36) by showing that it defines a contractive map for a suitable metric.

Uniqueness. Thus consider $(\mu_t, a(t))$ and $(\nu_t, b(t))$, any two solutions of (35), (36). Here their existence is assumed and their distance at time t satisfies

$$\begin{aligned} &\|(\mu_t, a(t)) - (\nu_t, b(t))\| \\ &\quad = d_{bL}(\mu_0 \circ T_{0,t}[a(.)], \nu_0 \circ T_{0,t}[b(.)]) + \|S_{t,0}[\mu_.]\, a(0) - S_{t,0}[\nu_.]\, b(0)\| \end{aligned} \tag{37}$$

In order to minimize the majoration of this discrepancy and provide a continuity relation versus initial conditions, we apply the triangular inequality to both terms of the right hand side and emphasize the parallel treatment of both particle and mode evolutions. This gives

$$\|S_{t,0}[\mu_.]\, a(0) - S_{t,0}[\nu_.]\, b(0)\| \leqslant d_1(t) + d_2(t) \tag{38}$$

$$d_{bL}(\mu_0 \circ T_{0,t}[a(.)], \nu_0 \circ T_{0,t}[b(.)]) \leqslant d_3(t) + d_4(t) \tag{39}$$

with

$$d_1(t) = \|S_{t,0}[\mu_.]\, a(0) - S_{t,0}[\mu_.]\, b(0)\| \tag{40}$$

$$d_2(t) = \|S_{t,0}[\mu_.]\, b(0) - S_{t,0}[\nu_.]\, b(0)\| \tag{41}$$

$$d_3(t) = d_{bL}(\mu_0 \circ T_{0,t}[a(.)],\, \nu_0 \circ T_{0,t}[a(.)]) \tag{42}$$

$$d_4(t) = d_{bL}(\nu_0 \circ T_{0,t}[a(.)],\, \nu_0 \circ T_{0,t}[b(.)]) \tag{43}$$

Considering the evolution of the wave-particles system in $\mathscr{F} \times \mathscr{X}$ requires estimating the distances $d_i(t)$. The sum $d_1(t) + d_3(t)$ represents the distance between the states obtained from different initial conditions evolving under the same constraints (forces and wave sources). The sum $d_2(t) + d_4(t)$ represents on the contrary the distance between two solutions at time t starting from the same initial conditions but evolving in different environments. We will first estimate one by one the distances $d_i(t)$.

Straightforward integration of (12) shows that

$$d_1(t) = d_1(0) = \|a(0) - b(0)\| \tag{44}$$

because the flow $S[\mu_.]$ is just a translation in $\mathscr{X}^M$.

To estimate d_2 we integrate (12) with the right hand sides given by $\mu_.$ and $\nu_.$:

$$d_2(t) = wq\beta' k^{-1} \left| \int_0^t \int_A e^{-ikx + i\omega_0 s}\, d(\mu_s - \nu_s)\, ds \right| \tag{45}$$

$$= 2wq\beta' k^{-1} \left| \int_0^t \int_A \frac{1 + i + e^{-ikx + i\omega_0 s}}{2}\, d(\mu_s - \nu_s)\, ds \right| \tag{46}$$

$$\leqslant \sqrt{2}\, \tau^{-1} \int_0^t d_{bL}(\mu_s, \nu_s)\, ds \tag{47}$$

In (47) the inequality uses the fact that $\alpha(1 + \cos(kx - \theta))/k \in \mathscr{D}$ and $\alpha(1 + \sin(kx - \theta))/k \in \mathscr{D}$ for any real θ, provided that $2\alpha \leqslant k$.

Concerning d_3, a direct calculation shows that $T_{t,0}[a(.)]$ is Lipschitz-continuous with respect to (x, v) with constant $\exp \int_0^t \gamma'[a(s)]\, ds$ so that Proposition 1 implies

$$d_3(t) \leqslant \left(\exp \int_0^t \gamma'[a(s)]\, ds \right) d_{bL}(\mu_0, \nu_0) \tag{48}$$

Finally, using the definition of $\mathscr{D}$ and applying triangular inequality gives

$$d_4(t) \leqslant \sup_A \|T_{t,0}[a(.)](x,v) - T_{t,0}[b(.)](x,v)\| \leqslant d_{41}(t) + d_{42}(t) \tag{49}$$

with

$$d_{41}(t) := \sup_A \left\| \int_0^t (G[a(s)]\; T_{s,0}[a(.)](x,v) - G[a(s)]\; T_{s,0}[b(.)](x,v))\, ds \right\| \tag{50}$$

$$\leqslant \int_0^t \gamma'[a(s)](d_{41}(s) + d_{42}(s))\, ds \tag{51}$$

$$d_{42}(t) := \sup_A \left\| \int_0^t (G[a(s)]\; T_{s,0}[b(.)](x,v) - G[b(s)]\; T_{s,0}[b(.)](x,v))\, ds \right\| \tag{52}$$

$$\leqslant \int_0^t \sup_A \|G[a(s)] - G[b(s)]\|\, ds \tag{53}$$

Definition (32) shows that

$$\begin{aligned} \|G[a(s)](x,v) - G[b(s)](x,v)\| &= \left| \frac{\alpha q \tau}{m} \beta'(a(s) - b(s))\, e^{ikx} \right| \\ &\leqslant \tau^{-1}\, \|a(s) - b(s)\| \end{aligned} \tag{54}$$

so that

$$d_{42}(t) \leqslant \tau^{-1} \int_0^t \|a(s) - b(s)\|\, ds \tag{55}$$

We now return to our symmetrical treatment of the particles and mode and define $D_1(t)$, a majorant of the sum $d_1(t) + d_3(t)$, and $D_2(t)$, a majorant of the sum $d_2(t) + d_4(t)$, as

$$D_1(t) := \|a(0) - b(0)\| + \left(\exp \int_0^t \gamma'[a(s)]\, ds \right) d_{bL}(\mu_0, \nu_0) \tag{56}$$

$$D_2(t) := d_2(t) + d_{41}(t) + d_{42}(t) \tag{57}$$

Then previous inequalities for the d_i ensure that

$$D_2(t) \leqslant \int_0^t \sqrt{2}\,\tau^{-1}\,D_1(s)\,ds + \int_0^t (\sqrt{2}\,\tau^{-1} + \gamma'[a(s)])\,D_2(s)\,ds \tag{58}$$

which Gronwall's inequality readily bounds by

$$D_2(t) \leqslant \int_0^t \sqrt{2}\,\tau^{-1}\,D_1(s)\,e^{\int_s^t (\sqrt{2}\,\tau^{-1} + \gamma'[a(u)])\,du}\,ds \tag{59}$$

The resulting complete estimate

$$\|(\mu_t, a(t)) - (\nu_t, b(t))\| \leqslant D_1(t) + D_2(t) \tag{60}$$

depends on two functions $\gamma'[a(s)]$ and $D_1(s)$. Note that $D_1(0) = \|(\mu_0, a(0)) - (\nu_0, b(0))\|$ and $D_2(0) = 0$. This ensures that estimate (60) does not grow faster than exponentially, with upper bound on its growth rate

$$C = \sqrt{2}\,\tau^{-1} + 2 \sup_{0 \leqslant s \leqslant t} \gamma'[a(s)] \tag{61}$$

which is bounded by a function of h_0 as discussed in Section 4. This proves that if a measure solution of the coupled system (35), (36) with given initial data in $(\mathscr{F} \times \mathscr{Z})$ exists then it is unique.

Remark. Our estimate for the growth rate C in the kinetic case is larger than the finite-N estimate for $|\mathscr{M}|$ in phase space. This is due to the fact that the distance d_{bL} makes no distinction between x-components and v-components, while estimates of Section 4 relied on treating these components of the phase space points separately to obtain (24).

Existence. Now it is sufficient to prove that (35), (36) admits a weakly continuous solution $(\mu_., a(.))$ in any arbitrary time interval $[0, T]$.

In the space $C_{\mathscr{F}}$ of weakly continuous time-dependent measures in $\mathscr{F}$, define, for any real $\eta > 0$, the distance d_η as

$$d_\eta(\mu_., \nu_.) := \sup_{0 \leqslant t \leqslant T} d_{bL}(\mu_t, \nu_t)\,e^{-\eta t} \tag{62}$$

Similarly, in the space $C_{\mathscr{Z}}$ of continuous functions on $\mathscr{Z}$ define the distance

$$\|a(.) - b(.)\|_\eta := \sup_{0 \leqslant t \leqslant T} \|a(t) - b(t)\|\,e^{-\eta t} \tag{63}$$

$(C_{\mathscr{F}}, d_\eta) \times (C_{\mathscr{X}}, \|.\|_\eta)$ is complete, since $(\mathscr{F}, d_{bL})$ and $(\mathscr{X}, \|.\|)$ are complete metric spaces.

Consider now the iteration scheme

$$\mu_t^{(n+1)} = \mu_0 \circ T_{0,t}[a(\,.\,)^{(n)}] \tag{64}$$

$$a(t)^{(n+1)} = S_{t,0}[\mu_{.}^{(n)}]\, a(0) \tag{65}$$

with $\mu_t^{(0)} = \mu_0$, $a(t)^{(0)} = a(0)$ and a given normalized energy $h_0 = h(\mu_0, a(0))$.

For any $t \in [0, T]$ and $n \geqslant 1$,

$$\begin{aligned} &\|a(t)^{(n+1)} - a(t)^{(n)}\| + d_{bL}(\mu_t^{(n+1)}, \mu_t^{(n)}) \\ &\qquad \leqslant \tau^{-1} \int_0^t (\sqrt{2}\, d_{bL}(\mu_s^{(n)}, \mu_s^{(n-1)}) + \|a(s)^{(n)} - a(s)^{(n-1)}\|\, e^{\int_s^t \gamma'[a(u)^{(n)}]\, du})\, ds \end{aligned} \tag{66}$$

where the first contribution is given by (41) and (47) and the second contribution follows from (43), (49), (51), (55) and Gronwall's inequality. A simple calculation then leads to

$$\begin{aligned} &d_\eta(\mu_{.}^{(n+1)}, \mu_{.}^{(n)}) + \|a(\,.\,)^{(n+1)} - a(\,.\,)^{(n)}\|_\eta \\ &\qquad \leqslant \tau^{-1}(\sqrt{2}\, \eta^{-1}\, d_\eta(\mu_{.}^{(n)}, \mu_{.}^{(n-1)}) + (\eta - C(h_0))^{-1}\, \|a(\,.\,)^{(n)} - a(\,.\,)^{(n-1)}\|_\eta) \end{aligned} \tag{67}$$

provided that η is chosen larger than a constant $C(h_0)$ of the form given in (61). Moreover for a suitable choice of η the iterative mapping can clearly be made contractive, so that the fixed point theorem applies ensuring the convergence of the scheme (64), (65) towards a unique fixed point in $(C_{\mathscr{F}}, d_\eta) \times (C_{\mathscr{X}}, \|.\|_\eta)$.

This completes the proof of the theorem. The corollary follows in a straightforward way, by noting that for any sequence $(\sigma_t^N, a^N(t))$ and kinetic solution $(\sigma_t, a^\infty(t))$ we can apply the result (18) for a constant C bounded by a function of $h(\sigma_0, a^\infty(0))$. As majorations are performed on a bounded time interval the uniform convergence follows trivially.

6. CONCLUSION

This work supports theoretically the use of full N-body dynamical schemes[10, 14, 20, 34, 35] to study the wave-particle interactions, as an alternative to kinetic-theory based models. This is particularly important for questions relating to the nature of irreversible evolution in finite-dimensional hamiltonian systems vs their kinetic counterpart which have continuous spectra.[9, 13, 20] However the regularity of the limit $N \to \infty$ is tempered by the rapid growth of the right hand side in the upper bound (18).

Conversely, note also that, in classical treatments of wave-plasma interactions, after obtaining the wave and plasma evolution using the kinetic approach self-consistently, one uses the resulting field evolution to compute the motion of particles neglecting any feedback by the latter on the fields (see, e.g., ref. 22). However the status of the "test" particles in these treatments is ambiguous at this stage (because their coupling with the fields is not self-consistent), so that a unified treatment of "test" particles and "kinetically-distributed" particles is desirable: the present results ensure the consistency of these classical approaches.

Our results are readily extended to the case of many waves and several particle species in several space dimensions.[33]

Finally, this work also identifies the fundamental cause of phase space mixing and approach to equilibrium in this many-body system: particles passing near the instantaneous saddle points associated with the modes undergo exponential dichotomy, with a divergence rate controlled by amplitudes $|z_j| = |a_j|$. This implies that the phase space regions where discrepancies between the kinetic description and the finite-N description show up most rapidly correspond to the neighbourhood of the "separatrices" associated with the envelopes in the particles" μ-space Λ, as was observed in numerical simulations for $M = 1$ by Guyomarc'h.[35]

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